Math 206A Lecture 3 Notes

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1 Generalized Helly's Theorem and Borsuk's Theorem

1.1 Corollaries of generalized Helly's theorem

We will show that Helly's theorem implies Borsuk's theorem in 2 dimensions.

Theorem 1.1 (generalized Helly). *Suppose $X_1, \dots, X_n, A$ are convex such that for all $i, j, k \in [n]$, X_i, X_j, X_k intersect some parallel translation of A. Then all X_i intersect some parallel translation of A.*

Corollary 1.1. *Let $z_1, \dots, z_n \in \mathbb{R}^2$ be such that for all i, j, k, z_i, z_j, z_k lie in some circle of radius 1. Then all z_i lie in some circle of radius 1.*

Proof. In generalized Helly, let X_i be $\{z_i\}$ and A be the circle of radius of 1. $\square$

Lemma 1.1. *Let $x, y, z \in \mathbb{R}^2$ be such that $|xy|, |xz|, |yz| \leq 1$. Then there exists a circle of radius $1/\sqrt{3}$ which covers them.*

Proof. There are 2 cases: ether the triangle xyz is acute or it is right or obtuse. If xyz is right or obtuse, then take R to be the midpoint of the longest edge. Then the points are contained in the circle of radius $1/2$ centered at R. If the triangle xyz is acute, let R be the center of the circle circumscribing the triangle. Let α be the angle xRz. Then $\alpha \geq 2\pi/3$. Then, since $|xz| \leq 1$, $|Rx| = |Rz| \leq 1/\sqrt{3}$. $\square$

Corollary 1.2. *For every $Z = \{z_1, \dots, z_n\} \subseteq \mathbb{R}^2$, if $|z_i z_j| \leq 1$ for all i, j, then there exists a circle of radius $1/\sqrt{3}$ which covers Z.*

Proof. Use the previous corollary and lemma. $\square$

1.2 Borsuk's theorem in 2 dimensions

Theorem 1.2 (Borsuk, d=2). *Suppose $X \subseteq \mathbb{R}^2$ is convex of diameter 1. Then $X = X_1 \cup X_2 \cup X_3$ such that* $\operatorname{diam}(X_i) < 1$.

We will start with an incorrect proof and then fix it in 3 places.

Proof. Put X in a circle of radius $1/\sqrt{3}$, and split the circle into 3 parts. Then the diameter of the circle is less than 1.

Error 1: Split a triangle into 3 parts. In this case, we have pieces with diameter ≤ 1, not < 1.

Error 2: The lemma needed a finite set, and X is infinite.

To fix error 1, take the circle and mark out two small regions on opposite sides of the circle. Then X cannot contain points of both sides at once. So X can still be covered by the truncated circle where we remove one of these regions. Now alter the partition by moving it $\varepsilon = 1/100$ away from the removed region. Then the diameters of the 3 parts of the circle are now all < 1.

To fix error 2, place X inside a polygon Q with diameter $1 + \delta$.

Here is error 3: We could have an issue in the fix of error 2 where we end up with pieces of diameter ≥ 1 because of the δ we added. However, if we take $\delta = 1/200$, we can avoid this situation because the pieces are not too big. In other words, the fixes of error 1 and error 2 must interact in some way. $\square$

1.3 Hadwiger's theorem

Borsuk's conjecture is false in general, but here is a result which says it is morally true.

Theorem 1.3 (Hadwiger)**.** *Borsuk's conjecture holds for smooth convex bodies.*

Proof. Step 1: (in $\mathbb{R}^3$ for simplicity) This is true when X is a ball of radius $1/2$. Inscribe a regular tetrahedron into the ball, and take a cone over each face. We can then partition the sphere into 4 cones with diameter < 1. So $X = C_1 \cup C_2 \cup C_3 \cup C_4$, which are cones from 0 over facets of the simplex.

Step 2: Since X is smooth, there is a tangent plane at every point on the boundary, giving us a normal vector at every point. Here is a lemma: If $\operatorname{diam}(X) = 1$, then for $|xy| = 1$, n_x and n_y are parallel. Now define $\gamma : \partial X \to S^{d-1}$ which takes $x \mapsto n_x$, the normal vector. Now let $Y_i = \gamma^{-1}(X_i)$. Then the partition from step 1 gives us a suitable partition of X. $\square$

What this theorem gives us is that for Borsuk's theorem to fail, we should really be looking at polytopes. So we only care about finitely many points, the extreme points of a convex polytope.